Laminar Flow in Strongly Curved Tubes

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The stable secondary flow which exists in incompressible laminar flow through curved tubes provides a potentially useful mechanism for increasing transport rates to the boundary relative to straight tubes. The coiled tube has been suggested as a useful geometry for certain chemical reactors by Koutsky and Adler (1964) because axial mixing is limited by the transverse flow (Erdogan and Chatwin, 1967; McConalogue, 1970; Nunge et al., 1972), and for reverse osmosis by Srinivasan and Tien (1971) and Nunge and Adams (1973), because concentration polarization is reduced. Furthermore, flow through porous media at Reynolds numbers above the Darcy regime has been suggested by Wright (1968) and Jones (1968) to have characteristics similar to those in curved tubes.

A number of experimental and analytical studies of laminar flow in curved tubes have been reviewed by Austin and Seader (1973). Most of the theoretical work has been done under the assumption of a large curvature ratio λ. With this assumption, the Dean number N_{De} is the only independent parameter of the flow.

The point of departure of the work to be described here is to remove the assumption of large λ and hence to provide macroscopic results over a wide range of parameters. Thus, the predictions are expected to find application in modeling the cardiovascular system and porous media wherein λ is likely to be small. The extension to small λ was accomplished by modifying the iterative Fourier series method of McConalogue and Srivastava (1968) for small N_{De} and the boundary layer approach of Itō (1969) for large N_{De}. The developments of these modifications in which λ and N_{De} are treated as independent parameters are lengthy but relatively straightforward and are given by Lin (1972).

RESULTS

To test the predictions of the modified techniques in the limiting case, results for $\lambda = 100$ were compared to existing results for $\lambda >> 1$. McConalogue (1970) presented an empirical correlation of the predictions of Q_c/Q_s versus N_{Dem}* obtained by McConalogue and Srivastava (1968). This correlation is shown on Figure 1 along with results obtained by the modified Fourier series method; clearly the curves for $\lambda >> 1$ and $\lambda = 100$ are in close agreement over this small Dean number range. The use of the modified iterative Fourier-series method was limited to small N_{De} because as N_{De} increased difficulty in obtaining convergence was encountered; a similar problem was experienced by McConalogue and Srivastava.

Topakoglu (1967) considered the independent effect of λ on the flow using a power series expansion. Difficulties with calculating higher order terms in his expansion limit the results to small values of N_{Dem}. Figure 1 shows a comparison of Q_c/Q_s for $\lambda = 2$ as predicted by the power series and Fourier series methods. These are in agreement for $N_{Dem} \lesssim 60$ ($N_{De} \lesssim 10$). A similar conclusion was reached by Austin and Seader (1973).

Itō (1969) demonstrated excellent agreement between the boundary layer solution for $\lambda >> 1$ and the experimental data of White (1929) for f_c/f_s as a function of N_{De}. The agreement between the predictions of the modified boundary layer technique for $\lambda = 100$ and those of Itō were also found to be excellent over the range of large N_{De}.

Figure 2 shows f_c/f_s versus N_{De} as predicted by the two methods used here along with some results obtained by Austin and Seader (1973) from a numerical solution of the equations of motion. For $\lambda = 100$, the present values agree with those of Austin and Seader. However, for smaller values of λ at larger constant N_{De}, the Austin-Seader predictions show f_c/f_s increasing with λ decreasing, while the present results exhibit the opposite trend. Austin and Seader argue that their curves show a correct trend since at small N_{De}, this same trend is indicated by the results of Topakoglu. However, the boundary layer model yields the same trend at smaller N_{De}; the curves for $\lambda = 2$ and 100 cross at $N_{De} \simeq 85$.

From physical arguments, one expects that for the same Reynolds number, it would require more energy to pump the same fluid through a more strongly curved tube (λ smaller). Both the Austin-Seader and the present results (see Figure 3) satisfy this criterion, and therefore the two cannot be differentiated on the basis of physical insight. Furthermore, a study of f_c/f_s versus N_{De} between $\lambda = 100$ and $\lambda = 2$ (Lin, 1972) indicates that $\lambda \lesssim 15$ is necessary before significant effects appear.

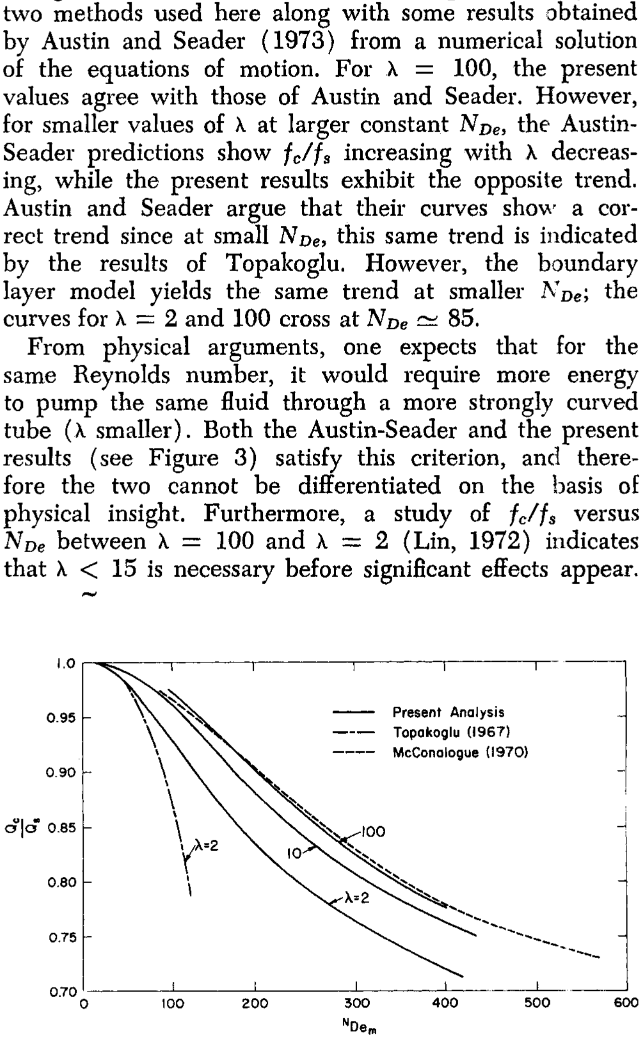

Fig. 1. The ratio of the volumetric flow rates in curve and straight tubes at the same axial pressure gradient Q_c/Q_s versus N_{Dem} with λ as a parameter.

* N_{De_m} is a form of the Dean number based on the Reynolds number in the straight tube. Under the assumption of equal axial pressure gradients, the Reynolds number in a curved tube is smaller.

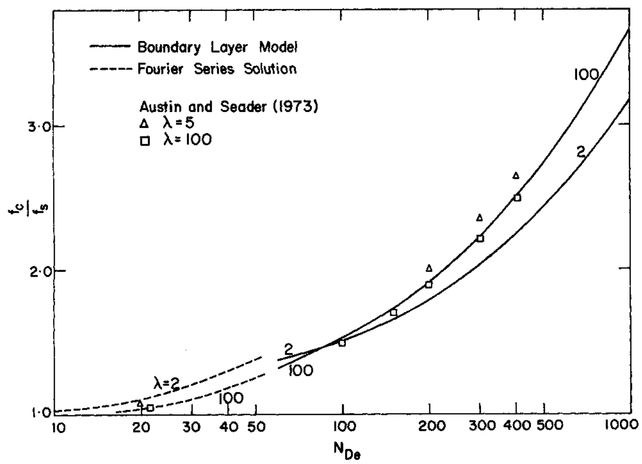

Fig. 2. The ratio of the friction factors in curved and straight tubes at the same Reynolds number f_c/f_s versus N_{De} with λ as a parameter.

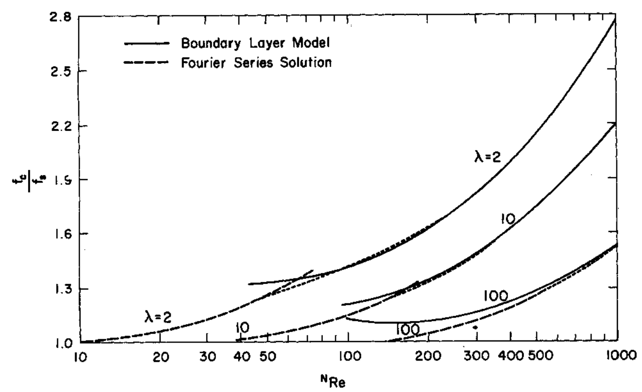

Fig. 3. The ratio of the friction factors in curved and straight tubes at the same Reynolds number f_c/f_s versus N_{Re} with λ as a parameter. The dotted lines give an estimate of the behavior in the region of uncertainty.

Since experimental data is not available for smaller values of λ, a determination between the Austin-Seader predictions and the present results cannot be made at present.

With solutions available for both small and large values of N_{De}, it is natural to attempt a matching of the two predictions and a determination of the lower limit of applicability of the boundary layer model. Itō claimed agreement between theory and experiment to N_{De} as small as 30. The situation at smaller λ is more complicated because f_c/f_s depends on both λ and N_{De} and experimental data is not available.

In the usual boundary layer approach, the model is expected to fail at the Reynolds number for which the boundary layer is a significant fraction of the transverse dimension. However, this is not a usual boundary layer model since the boundary layer thickness in the curved tube depends upon λ and N_{Re}, and thus the model is expected to fail at different values of N_{Re} for each value of λ.

Data from Figure 2 has been replotted as a function of N_{Re} in Figure 3 and an attempt made to join the two solutions smoothly. It appears that for $\lambda = 100$, the boundary layer and Fourier series results can be smoothly joined by extrapolating the curve for small N_{Re} to $N_{Re} \simeq 1000$. At constant N_{Re}, calculations show that the boundary layer thickness decreases with λ decreasing and thus the inference is that the boundary layer model should be valid to smaller N_{Re} at smaller λ. If it is arbitrarily assumed that the dimensionless boundary layer thickness is the same for all λ when the boundary layer model fails, then the lower limits for N_{Re} are 355 and 230 for $\lambda = 10$ and 2, respectively. Using these values, the dotted lines on Figure 3 were used to join the two solutions smoothly. The differences between the predictions of the two methods over the range of uncertainty shown by the dotted lines are small.

ACKNOWLEDGMENT

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NOTATION

a = inside tube radius
f_c/f_s = friction factor ratio for the same Reynolds number in curved and straight tubes
N_{De} = Dean number, $N_{Re}\lambda^{-1/2}$
N_{Dem} = modified Dean Number, $4\sqrt{2}\,N_{Re_s}\,\lambda^{-1/2}$
N_{Re} = Reynolds number based on the average velocity in a curved tube, $2a\,W_{mc}/\nu$
N_{Re_s} = Reynolds number based on the average velocity in a straight tube, $2a\,W_{ms}/\nu$
Q_c/Q_s = volumetric flux ratio for the same axial pressure gradient in curved and straight tubes
R = radius of curvature
W_m = average velocity

Greek Letters

λ = curvature ratio, R/a
ν = kinematic viscosity

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